THE EFFECT OF A LONGITUDINAL POSITIVE PRESSURE GRADIENT ON THE CRITICAL INJECTION PARAMETER

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On the basis of conclusions from the asymptotic theory of the boundary layer we have derived a theoretical relationship between the magnitude of the critical injection parameter and the positive longitudinal pressure gradient, as well as velocity profiles at the point of displacement for these conditions. Results of experimental work are in satisfactory agreement with theoretical data.

The possibility of the onset of critical injection for a turbulent boundary layer on a permeable plate is demonstrated theoretically in [1,2] and confirmed experimentally in [3,4]. Here an attempt is made to analyze the magnitude of the critical injection parameter b_{cr} as a function of the longitudinal positive pressure gradient for a turbulent boundary layer.

As is well known,

$$b = \bar{j}_w \frac{2}{Cf_0}, \quad Cf_0 = \frac{2}{(2.5 \ln Re^{**} + 3.8)^2},$$

$$\bar{j}_w = \frac{\rho_w W_w}{\rho_0 W_0}, \quad Re^{**} = \frac{W_0 \delta^{**}}{\nu_0}. \tag{1}$$

It follows from [1] that on injection of a homogeneous gas, for an isothermal turbulent boundary layer, when $Re^{**} \to \infty$ and $dp/dx \geq 0$, the following relationship is valid:

$$\frac{2}{3} < \int_0^1 \frac{d\omega}{\sqrt{\psi \frac{\bar{\tau}}{\tau_0}}} \leqslant 1, \tag{2}$$

where

$$\psi \frac{\bar{\tau}}{\tau_0} = \psi + (\lambda_0 \zeta + b\omega) f(\zeta); \tag{3}$$

$$\lambda_0 = -\frac{\delta}{\delta^{**}} \frac{2}{Cf_0} f; \quad f = \frac{\delta^{**}}{W_0} \frac{dW_0}{dx}. \tag{4}$$

Equation (2) with consideration of (3) for the conditions of critical injection ($\psi = 0$) can be written approximately in the form

$$\int_0^1 \frac{d\omega}{\sqrt{(\lambda_0 \zeta + b_{cr}\omega) f(\zeta)}} \approx 1. \tag{5}$$

First let us examine the case in which $f(\zeta) = 1$ and $\zeta = \omega^2$. The possibility of assuming $f(\zeta) = 1$ is considered in [1,2].

With regard to the second assumption we can make the following comment. If there is no transverse flow of matter ($b = 0$), Eq. (5) corresponds to the conditions of boundary-layer separation due to a positive pressure gradient. The velocity profile at the point of separation, as is well known, is approximately given by the equation $\omega = \zeta^{0.5}$. This relationship may be derived from the equation of motion near the wall in the absence of a lateral flow of matter and if molecular viscosity forces are neglected.

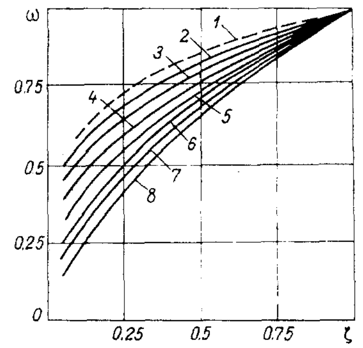

Fig. 1. Velocity distribution in turbulent core of boundary layer in the first approximation of $Re^{**} = 4000$ (Eq. (12)). Curves 1, 2, 3, 4, 5, 6, 7, and 8 have been found for values of $\lambda_0 = 0$, 5, 10, 15, 20, 25, 30, and 35, respectively.

As $\lambda_0 \to 0$ the value of b_{cr}, defined in (5), tends toward 4 which agrees with [1]. Thus the assumption $\zeta = \omega^2$ is completely acceptable for the limit conditions.

In view of the above, (5) transforms to

$$\int_0^1 \frac{d\omega}{\sqrt{\lambda_0 \omega^2 + b_{cr}\omega}} \approx 1,$$

which after integration yields

$$\frac{1}{\sqrt{\lambda_0}} \ln \left[\frac{2\sqrt{\lambda_0}\sqrt{\lambda_0 + b_{cr}} + 2\lambda_0 + b_{cr}}{b_{cr}}\right] \approx 1. \tag{6}$$

Solving (6) for b_{cr}, we obtain

$$b_{cr} = \frac{4\lambda_0 \exp\sqrt{\lambda_0}}{[\exp(\sqrt{\lambda_0}) - 1]^2}. \tag{7}$$

For sufficiently large λ_0 formula (7) can be simplified to

$$b_{cr} = \frac{4\lambda_0}{\exp\sqrt{\lambda_0}}.$$

Formula (7) satisfies the limit conditions: as $\lambda_0 \to 0$, $b_{cr} \to 4$ and as $\lambda_0 \to \infty$, $b_{cr} \to 0$. To clarify

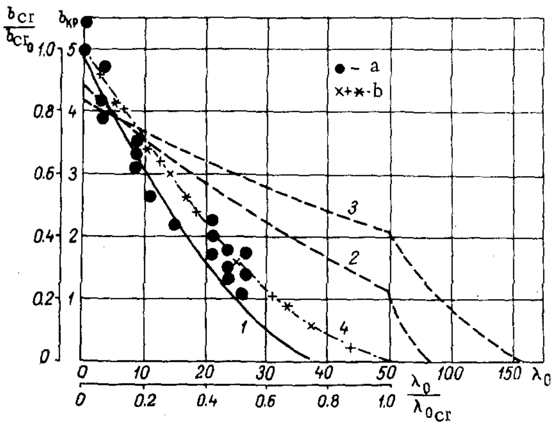

Fig. 2. Dependence of critical injection parameter on longitudinal positive pressure gradient for coordinates b_{cr}, λ_0(curves 1, 2, and 3 correspond to Eq. (13) for Re** = 4000, 10^6, 10^8, respectively; a shows experimental points), and for the coordinates b_{cr}/b_{cr0}; λ_{0cr} (curve 4 according to Eq. (14); b are points derived by recalculation from curves 1, 2, and 3).

the latter circumstance it is appropriate that we recall that according to [1], on an impermeable plate (b = 0), boundary-layer separation occurs if

$$\lambda_{0\,cr} = 0.062\,\frac{2}{Cf_0}. \tag{8}$$

With (7) we find the expression for the velocity profile at the displacement point for $b_{cr} \neq 0$, $\lambda_0 \neq 0$, using the fundamental premises in analogy with [2] for this:

$$\tau = \rho l^2 \left(\frac{dW}{dy}\right)^2, \quad \frac{l}{\delta} = 0.4\zeta\sqrt{\bar{\tau}_0}$$

or

$$\bar{\tau} = \frac{2}{Cf}\left(0.4\,\zeta\sqrt{\bar{\tau}_0}\,\frac{d\omega}{d\zeta}\right)^2. \tag{9}$$

Expression (3) corresponds to

$$\frac{\bar{\tau}}{\bar{\tau}_0} = 1 + \frac{(\lambda\zeta + b_1\omega)}{1+2\zeta}, \tag{10}$$

where

$$\lambda = -\frac{\delta}{\delta^{**}}\,\frac{2}{Cf}\,f; \quad b_1 = \frac{\rho_w W_w}{\rho_0 W_0}\,\frac{2}{Cf}.$$

Having substituted (10) into (9) we obtain

$$\left[\frac{Cf}{2} + \left(-\frac{\delta}{\delta^{**}}\,f\right)\zeta + \bar{j}_w\,\omega\right] = \left(0.4\,\zeta\,\frac{d\omega}{d\zeta}\right)^2.$$

Since we are dealing with the case of a displaced boundary layer, $Cf/2 \to 0$, and we finally have

$$\left(\frac{d\omega}{d\zeta}\right)^2 - \frac{B\omega}{\zeta^2} = \frac{A}{\zeta} \quad (\zeta = 1,\ \omega = 1), \tag{11}$$

where

$$B = \frac{b_{cr}\,Cf_0}{2\cdot 0.16}; \quad A = \frac{\lambda_0\,Cf_0}{2\cdot 0.16}.$$

With consideration of (7), Eq. (11) was solved on a computer for Re** = 4000, 10^6, 10^8. The derived families of velocity profiles are approximated by the power function (Fig. 1)

$$\omega = \zeta^q, \tag{12}$$

where

$$q = \frac{0.04\,\lambda_0 + 1}{4}; \quad q = \frac{0.05\,\lambda_0 + 1}{10}; \quad q = \frac{0.058\,\lambda_0 + 1}{20}$$

where Re** = 4000, 10^6, 10^8, respectively.

Having substituted (12) into (5), for $f(\zeta) = [1/(1+2\zeta)] \neq 1$, we obtain an expression for the second approximation of the function $b_{cr} = f(\lambda)$

$$\int_0^1 \frac{d\omega}{\sqrt{(\lambda_0\omega^{1/q} + b_{cr}\,\omega)\,\dfrac{1}{2\omega^{1/q}+1}}} \approx 1. \tag{13}$$

The results of the solution of (13) on a computer for Re** = $4\cdot 10^3$, 10^6, 10^8 are given in Fig. 2.

All of these solutions may be approximated rather satisfactorily by the formula

$$\frac{b_{cr}}{b_{cr\,0}} = \left(1 - \frac{\lambda_0}{\lambda_{0\,cr}}\right)^{1.55}, \tag{14}$$

where $b_{cr\,0}$ is the value of the critical injection parameter for $\lambda_0 = 0$; $\lambda_{0\,cr} = 0.062\ 2/Cf_0$ is the parameter characterizing the separation of the boundary layer as a result of a longitudinal positive pressure gradient [1].

Strictly speaking, the calculation of the functions $b_{cr} = f(\lambda)$ from (13) is possible only for those values of λ_0 which yield q = 0.5 in (12), since q > 0.5 seems to correspond to the separation zone. However, the solution for (12) and (13) without the limiting condition $q \leq 0.5$ for the cases $b_{cr} = 0$ yields values of λ_0 virtually satisfying (8).

A second approximation for the velocity profiles in the boundary layer at the point of displacement was carried out with consideration of (14). For these purposes we use the equation

$$\left(\frac{d\omega}{d\zeta}\right)^2 - \frac{B\omega}{\zeta^2(1+2\zeta)} = \frac{A}{\zeta(1+2\zeta)}, \tag{15}$$

which was derived in analogy with (11), but in the assumption that

$$f(\zeta) = \frac{1}{1+2\zeta} \neq 1.$$

It is interesting to note that (15) when B = 0, A ≠ 0 (absence of injection) yield the earlier derived relationship $\omega = \varphi(\zeta)$ at the point of boundary-layer separation [1]:

$$\omega = \frac{\ln(2\sqrt{2}\sqrt{2\zeta^2+\zeta}+4\zeta+1)}{\ln(2\sqrt{6}+5)} \approx \zeta^{0.43}.$$

For the conditions A = 0, B ≠ 0 formula (15) yields a velocity profile at the displacement point in the absence of a longitudinal pressure gradient

$$\omega = \left[\frac{2.5\sqrt{b_{cr}}}{2}\sqrt{\frac{Cf_0}{2}}\left(\ln\frac{\sqrt{1+2\zeta}-1}{\sqrt{1+2\zeta}+1} - \ln\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)+1\right]^2. \qquad (16)$$

Formula (16) differs from the function $\omega = \varphi(\zeta)$ derived earlier at the displacement point [2]:

$$\omega = \left(1 + 2.5\sqrt{\frac{Cf_0}{2}}\ln\zeta\right)^2 \frac{b_{cr}}{4}. \qquad (17)$$

This is explained by the fact that in the derivation of (15), and consequently, in the derivation of (16), the function $f(\zeta)$ was assumed as not equal to unity. As is demonstrated in [2] $f(\zeta) \to 1$, if $Re^{**} \to \infty$. Therefore it is natural to assume in (17) a value of $b_{cr} = 4$ (as the limit value of b_{cr} as $Re^{**} \to \infty$).

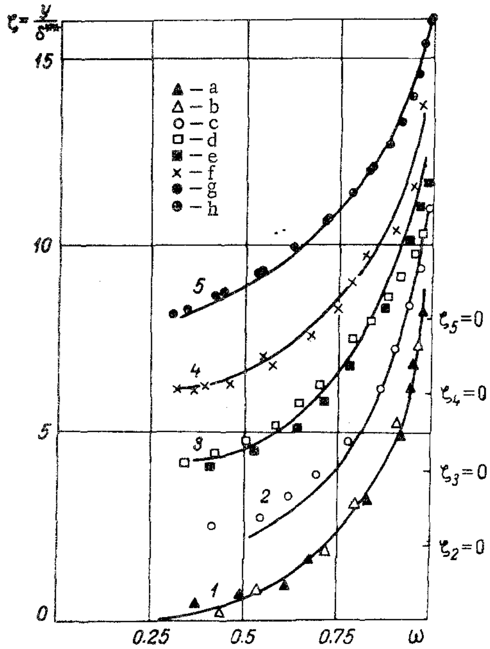

Fig. 3. Velocity distribution in turbulent core of boundary layer at displacement point. The curves represent the second approximation of Eqs. (14) and (15): 1) $\lambda_0 = 0$, $Re^{**} = 1300$; 2) 4 and 2000; 3) 9 and 3000; 4) 20 and 4000; 5) 25 and 3500; the experimental points: a) $\lambda_0 = 0$, $Re^{**} = 860$; b) 0 and 1400; c) 4 and 2000; d) 9 and 3000; e) 11 and 2000; f) 20 and 4000; g) 24 and 4300; h) 26 and 3520.

It can be assumed that $f(\zeta) = [1/(1 + 2\zeta)]$ to some extent gives consideration to the effect of Re^{**} on the velocity profile. We assume in this case that the values of b_{cr} in (16) must be functions of Re^{**}. Indeed, if the values of b_{cr} are found with consideration of the finiteness of Re^{**} [2] or experimentally [3] or

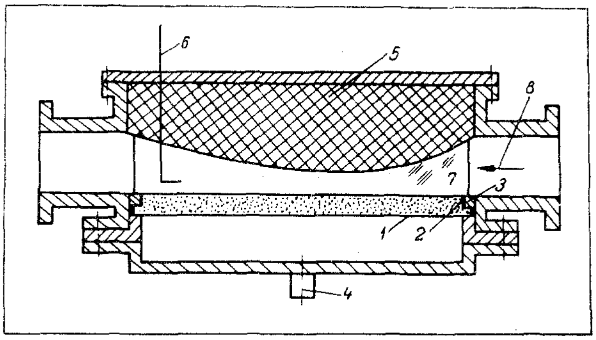

Fig. 4. Diagram of test section: 1) porous plate; 2) sealing (epoxy resin); 3) clamp frame; 4) supply of injected liquid; 5) insert; 6) Pitot tube; 7) viewing window; 8) bulk flow.

substituted into (16), the latter will be in satisfactory agreement with the experimental velocity profile at the displacement point (Fig. 3, curve 1). It is interesting to note that the velocity profiles obtained from (15), but with consideration of (7), virtually do not differ from the joint solution of (14) and (15).

To verify the derived results we carried out an experiment to determine the velocity profiles in the boundary layer at the point of displacement, as well as the magnitudes of the critical injection parameter. The experiments were carried out on the hydraulic installation described in detail in [3]. The method of determining b_{cr} is based on the chemical interaction of the main flow (a solution of hydrochloric acid) with the injected liquid (a weak alkaline solution stained with phenolphthalein). At the beginning of the critical injection a clearly visible thin film of injected liquid appears at the surface of the porous plate.

The test section is shown schematically in Fig. 4. The maximum dimensions of its flowthrough section are 30 × 40, the length of the porous plate is 200 mm, and the length of the preconnected section is 60 mm. To obtain the longitudinal pressure gradient we placed inserts in the section, these having been sealed against possible side leakages. The static pressure in the flow along the length of the channel was measured through samples in the side wall and by means of a special sensor which could be moved along the channel. This sensor was made of a 3 mm diameter tube situated parallel to the channel axis. The cylindrical surface of the tube had an orifice for sampling of static pressure. The length of the sensor was chosen so that the ends of the tube extended beyond the limits of the working section as the sensor was moved longitudinally. The distribution of static pressure along the length of the channel was linear, thus making it possible exactly to determine the value of dP/dx.

The quantity Re^{**}, needed to determine b_{cr} from (1), was found from experiment and reached 4000.

For the magnitude of the longitudinal pressure gradient to exert no influence on the uniformity of injection along the length of the working section, the porous plate was selected so as to exhibit great hydraulic resistance (in our experiments $\Delta P > 1$ atm). The plate was made of porous polymethylmethacrylate L-3, produced by sintering. The plate was covered on top with two layers of white polycaprolactam [caprone] fabric to impart the properties of opacity to the specimen. The dimensions of the active porous surface were 200×6 mm.

The measurements of the velocity profiles at three sections along the width of the porous plate (axis, ±1.5 mm) showed the presence of a flow core.

The experimental values of b_{cr} are compared in Fig. 2 with the theoretical data for Re** = 4000 (curve 1). The experimental values of Re** are given in the key to Fig. 3 for corresponding values of λ_0. As we can see from the graph, the experimental values of b_{cr} not only confirm the relative formula (14), but are also in satisfactory agreement with the absolute values of b_{cr} found from (13).

The velocity profiles calculated from (14) and (15) and measured experimentally are shown in Fig. 3 for the displacement points. These data are also in satisfactory agreement.

NOTATION

Here ρ_W, W_W, ρ_0, and W_0 are the density and velocity at the wall and external boundary layer, respectively; δ^{**} is the momentum thickness; ν_0 is the kinematic viscosity; ω is the relative velocity; ξ is the dimensionless coordinate; ψ is the relative friction factor; $\bar{\tau}$ and $\bar{\tau}_0$ are the relative shear stresses under conditions of interest and with no disturbing factors; l is the mixing length; $\varkappa$ is the turbulence constant; Cf is the friction factor; b_{cr} is the critical injection parameter; λ, λ_0, and f are the parameters characterizing the longitudinal pressure gradient.

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